

BADLY APPROXIMABLE S -NUMBERS AND ABSOLUTE SCHMIDT GAMES

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ABSTRACT. Let K be a number field, let S be the set of all normalized, non-conjugate Archimedean valuations of K , and let $K_S = \prod_{v \in S} K_v$ be the Minkowski space associated with K . We strengthen recent results of [ESK10] and [EGL13] by showing that the set of badly approximable elements of K_S is \mathcal{H} -absolute winning for a certain family of subspaces of K_S .

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1. INTRODUCTION

In the classical theory of Diophantine approximation, a number $x \in \mathbb{R}$ is called *badly approximable* if there exists $c = c_x > 0$ such that for every $p \in \mathbb{Z}$, $q \in \mathbb{N}$:

$$\left| x - \frac{p}{q} \right| > \frac{c}{q^2}, \text{ or equivalently, } |qx - p| > \frac{c}{q}. \quad (1.1)$$

The set of badly approximable numbers is small in that its Lebesgue measure is 0. Nevertheless, this set is *thick*, which means that its intersection with any open set in \mathbb{R} has full Hausdorff dimension. In fact, the intersection of any countable collection of translations of this set is still thick. This remarkable result was first proved by W. Schmidt [Sch66] by showing that the set of badly approximable numbers is a winning set of an infinite topological game, which will be called *Schmidt game*. In Schmidt game, two players (Alice and Bob) alternatively choosing nested balls and Alice's goal is to steer the intersection point to the target set, which is called a *winning set* of the game if Alice has a winning strategy (see §4.1). Inspired by Schmidt, C. McMullen [McM10]

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introduced a variation of Schmidt game, in which Alice can only delete neighborhoods of points, and proved that the set of badly approximable numbers is winning on this game. McMullen's version of the game is called *absolute game*, and its winning sets are called *absolute winning*. Absolute winning is a stronger property than winning and is preserved by quasi-symmetric maps (see §4.2).

Our goal is to generalize the above results into the setting of algebraic number fields. To be more precise, let K be a number field of degree d and let S be the set of all normalized, non-conjugate Archimedean valuations of K . If v is an element of S , the corresponding embedding of K into its completion K_v is denoted by ι_v . We will replace \mathbb{R} by the *Minkowski space* associated with K , defined by:

$$K_S := \prod_{v \in S} K_v \cong K \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^d; \quad (1.2)$$

elements $\mathbf{x} = (x_v)_{v \in S} \in K_S$ will be referred to as *S-numbers*. The diagonal embedding of K into K_S is denoted by ι_S :

$$\iota_S : K \rightarrow K_S, r \mapsto (\iota_v(r))_{v \in S}.$$

We also equip K_S with the K_S -valued inner product:

$$\mathbf{x} \cdot \mathbf{y} = (x_v y_v)_{v \in S}$$

and the sup norm

$$\|\mathbf{x}\| = \max_{v \in S} |x_v|$$

for every $\mathbf{x} = (x_v)_{v \in S}$ and $\mathbf{y} = (y_v)_{v \in S}$ in K_S .

We let \mathcal{O} be the ring of integers of K ; note that $\iota_S(\mathcal{O})$ is a lattice in K_S .

Following [EGL13], we will say that $\mathbf{x} \in K_S$ is *badly approximable* if there exists $c > 0$ such that for every $p \in \mathcal{O}$, $q \in \mathcal{O} \setminus \{0\}$:

$$\|\iota_S(q) \cdot \mathbf{x} + \iota_S(p)\| = \max_{v \in S} |\iota_v(q)x_v + \iota_v(p)| > c \left(\max_{v \in S} |\iota_v(q)| \right)^{-1} = c \|\iota_S(q)\|^{-1}. \quad (1.3)$$

or equivalently,

$$\inf\{\|\iota_S(q)\| \cdot \|\iota_S(q) \cdot \mathbf{x} + \iota_S(p)\| : p, q \in \mathcal{O}, (p, q) \neq 0\} > 0. \quad (1.4)$$

Let us denote the set of badly approximable S -numbers by \mathbf{BA}_K . In particular, when $K = \mathbb{Q}$, (1.3) is the same as (1.1), and we come back to the classical case. Measure-wise, \mathbf{BA}_K is small, that is, it can be shown to have Lebesgue measure zero, see [EGL13] or Remark 2.7 below. Explicit examples of badly approximable S -numbers for arbitrary number field K were constructed by Burger [Bur92], and when K is real quadratic or totally complex quartic, Hattori [Hat07] showed that \mathbf{BA}_K is uncountable. Note that those authors used different definitions of badly approximable numbers, but their examples in fact belong to \mathbf{BA}_K . The first result regarding the winning property of \mathbf{BA}_K beyond the classical case was proved by Esdahl-Schou and Kristensen in [ESK10] as follows:

Theorem 1.1 ([ESK10, Lemma 4]). *Let K be an imaginary quadratic number field with class number 1, i.e., $K = \mathbb{Q}(\sqrt{-D})$, where $D \in \{1, 2, 3, 7, 11, 19, 43, 67, 163\}$, and $Y \subseteq \mathbb{C} \cong K_S$ be a compact set supporting an Ahlfors regular measure μ . Then $\mathbf{BA}_K \cap Y$ is winning for the Schmidt game playing on Y .*

We recall that μ is said to be *Ahlfors regular* if there exist constants $a, b, \delta, r_0 > 0$ such that for any $z \in Y$ and any $0 < r \leq r_0$,

$$ar^\delta \leq \mu(\mathbf{B}(z, r)) \leq br^\delta, \quad (1.5)$$

where $\mathbf{B}(z, r)$ is the closed ball centered at z of radius r . It follows that $Y = \text{supp } \mu$ has Hausdorff dimension δ , and one knows, see e.g. [ESK10, Lemma 3], that any subset of Y which winning on Y is thick.

A few years later, Einsiedler et al. [EGL13] considered an arbitrary number field K and showed that the intersections of \mathbf{BA}_K with certain smooth curves in K_S are winning. The following

notation is needed to state their result: for a subset $T \subseteq S$, denote by $T_{\mathbb{R}}$ and $T_{\mathbb{C}}$ the sets of valuations in T whose corresponding K -completions are isomorphic to \mathbb{R} or \mathbb{C} respectively:

$$T_{\mathbb{R}} := \{v \in T : K_v \cong \mathbb{R}\} \quad \text{and} \quad T_{\mathbb{C}} := \{v \in T : K_v \cong \mathbb{C}\}.$$

Theorem 1.2 ([EGL13, Theorem 1.1]). *Let $\phi : [0, 1] \rightarrow K_S$ be a continuously differentiable map. For any $x \in [0, 1]$, define*

$$T(x) := \{v \in S : \phi'_v(x) \neq 0\}.$$

Assume that for all but finitely many $x \in [0, 1]$ we have

$$\#T(x)_{\mathbb{R}} + 2(\#T(x)_{\mathbb{C}}) > \frac{d}{2}.$$

Then $\phi^{-1}(\mathbf{BA}_K)$ is winning, and hence $\mathbf{BA}_K \cap \phi([0, 1])$ is thick in $\phi([0, 1])$.

In particular it follows that the set \mathbf{BA}_K is itself thick. The goal of this paper is to prove a stronger property of the set \mathbf{BA}_K , from which the conclusions of Theorem 1.1 and Theorem 1.2 follow. Namely, following [McM10, BFK⁺12, FSU13] in §4 we describe so-called \mathcal{H} -absolute game on a subset of a complete metric space, where \mathcal{H} is a collection of subsets of the space. In this game, compared with McMullen's version, Alice deletes neighborhoods of sets from \mathcal{H} instead of points. Winning sets of this game are also winning for the regular Schmidt's game, and, moreover, the winning properties are inherited by certain nice subsets of the ambient space.

To state our main result we need to introduce some notation. If T is a subset of S and $\mathbf{x} \in K_S$, we let

$$\mathcal{L}(\mathbf{x}, T) = \{\mathbf{y} \in K_S : y_v = x_v \ \forall v \in T\} \tag{1.6}$$

be the affine subspace of K_S passing through \mathbf{x} and orthogonal to coordinate directions corresponding to $v \in T$. Then let \mathcal{H}_K be the following family of affine subspaces:

$$\mathcal{H}_K = \left\{ \mathcal{L}(\mathbf{x}, T) : \mathbf{x} \in \iota_S(K), T \subseteq S \text{ with } \#T_{\mathbb{R}} + 2(\#T_{\mathbb{C}}) > \frac{d}{2} \right\}. \tag{1.7}$$

Our choice of \mathcal{H}_K will be explained in §5. In particular, we will show in Proposition 5.5 that for any $\mathcal{L} \in \mathcal{H}_K$,

$$\mathcal{L} \cap \mathbf{BA}_K = \emptyset,$$

so the collection \mathcal{H}_K is, in some sense, optimal.

Here is our main theorem:

Theorem 1.3. *\mathbf{BA}_K is \mathcal{H}_K -absolute winning.*

As a consequence of the main theorem, at the end of §4.2 we will prove the following corollary, which will imply Theorem 1.2.

Corollary 1.4. *Let $\phi : [0, 1] \rightarrow K_S$ be a C^1 curve such that for all but countably many $x \in [0, 1]$, we have*

$$\#T(x)_{\mathbb{R}} + 2(\#T(x)_{\mathbb{C}}) > \frac{d}{2}.$$

Then $\phi^{-1}(\mathbf{BA}_K)$ is absolute winning.

Moreover, notice that for any $\mathcal{L} = \mathcal{L}(\mathbf{x}, T) \in \mathcal{H}_K$,

$$\dim_{\mathbb{R}}(\mathcal{L}) = d - (\#T_{\mathbb{R}}) - 2(\#T_{\mathbb{C}}) = (\#(S \setminus T)_{\mathbb{R}}) + 2(\#(S \setminus T)_{\mathbb{C}}) < \frac{d}{2}. \tag{1.8}$$

So when $d = 1, 2$, or when K is totally complex and $d = 4$, \mathcal{H}_K consists of only points (0-dimensional subspaces), and hence the \mathcal{H}_K -absolute game coincides with the absolute game defined in [McM10]. Thus we have the following extension of McMullen's result:

Corollary 1.5. *If $d = 1, 2$, or K is totally complex quartic number field, then \mathbf{BA}_K is absolute winning.*

In particular, this implies that in those cases $\phi^{-1}(\mathbf{BA}_K)$ is winning for any smooth curve $\phi : [0, 1] \rightarrow K_S$, regardless of directions in which its derivative is zero. Moreover, in view of the inheritance property (see §4.2 for more details), we obtain the following corollary strengthening Theorem 1.1:

Corollary 1.6. *Let K be an imaginary quadratic number field, and $Y \subseteq \mathbb{C} \cong K_S$ be the support of an Ahlfors regular measure, then \mathbf{BA}_K is absolute winning on Y .*

The structure of this paper is as follows. In §2 we discuss the setting of Diophantine approximation in number fields in more details. Our main tools, the height function and Dani's correspondence (Proposition 2.6), are described in §2.2. In §3 we define \mathcal{H} -diffuse sets, and in §4 we introduce Schmidt game and the \mathcal{H} -absolute game and derive Corollaries 1.4 and 1.6 from the main theorem. The proof of Theorem 1.3 is given in §5. In Appendix A, we will show that the badly approximable S -numbers considered by Hattori [Hat07] are in fact the same as \mathbf{BA}_K . And finally, in Appendix B we give the proofs of basic properties of \mathcal{H} -absolute winning sets.

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2. DIOPHANTINE APPROXIMATION IN NUMBER FIELDS

2.1. Dirichlet Theorem. As in classical theory of Diophantine approximation, the justification of our definition of badly approximable S -numbers (1.3) is the following version of Dirichlet's Theorem:

Proposition 2.1 (Strong Dirichlet Theorem). *There exists a constant $C = C_K > 0$ depending only on K , such that for every $\mathbf{x} \in K_S$ and for every $Q > 0$, there exists $p \in \mathcal{O}$, $q \in \mathcal{O} \setminus \{0\}$ with*

$$\|\iota_S(q) \cdot \mathbf{x} + \iota_S(p)\| \leq CQ^{-1} \quad \text{and} \quad \|\iota_S(q)\| \leq Q. \quad (2.1)$$

For more notations from algebraic number theory, let us denote the *local degree* at $v \in S$ by:

$$d_v := [K_v : \mathbb{Q}_v] = [K_v : \mathbb{R}] = \begin{cases} 1 & , v \in S_{\mathbb{R}} \\ 2 & , v \in S_{\mathbb{C}} \end{cases}.$$

And for $a \in \mathcal{O}$, (the absolute value of) the field norm of a is defined by:

$$|N(a)| = \prod_{v \in S} |\iota_v(a)|^{d_v}.$$

To prove Proposition 2.1, we will make use of the following (special case of) lemma of Burger [Bur92]:

Lemma 2.2 ([Bur92, Lemma 5.1]). *Let $\mathbf{x} \in K_S$ and for each $v \in S$, let $0 < \varepsilon_v < 1 \leq \delta_v$ such that*

$$\prod_{v \in S} (\delta_v \varepsilon_v)^{d_v} \geq \left(\frac{2}{\pi}\right)^{2(\#S_{\mathbb{C}})} |D_K|,$$

where D_K is the discriminant of K . Then there exists $p, q \in \mathcal{O}$, $q \neq 0$, satisfying

$$|\iota_v(q)x_v + \iota_v(p)| \leq \varepsilon_v \quad \text{and} \quad |\iota_v(q)| \leq \delta_v, \quad \text{for all } v \in S.$$

Proof of Proposition 2.1. Let

$$C = \left(\left(\frac{2}{\pi}\right)^{2(\#S_{\mathbb{C}})} |D_K| \right)^{1/d},$$

then by applying Lemma 2.2 with

$$\delta_v = Q \quad \text{and} \quad \varepsilon_v = CQ^{-1} \quad \text{for all } v \in S,$$

we can find $p \in \mathcal{O}$, $q \in \mathcal{O} \setminus \{0\}$ satisfying (2.1). \square

Theorem 2.3 (Weak Dirichlet Theorem). *There is a constant $C = C_K > 0$ depending only on K , such that for every $\mathbf{x} \in K_S$, there are infinitely many $p, q \in \mathcal{O}$, $q \neq 0$ satisfying:*

$$\|\iota_S(q) \cdot \mathbf{x} + \iota_S(p)\| \leq C \|\iota_S(q)\|^{-1}. \quad (2.2)$$

Proof. Note that

$$\|\iota_S(q) \cdot \mathbf{x} + \iota_S(p)\| = 0 \iff \mathbf{x} = \iota_S\left(\frac{p}{q}\right).$$

If $\mathbf{x} \in \iota_S(K)$, then the existence of infinitely many p, q is obvious. Otherwise, for every $p, q \in \mathcal{O}$ with $q \neq 0$,

$$\|\iota_S(q) \cdot \mathbf{x} + \iota_S(p)\| > 0.$$

Moreover, for every $Q > 0$,

$$\#\{q \in \mathcal{O} : \|\iota_S(q)\| \leq Q\} < \infty.$$

Hence, by letting $Q \rightarrow \infty$ and applying Proposition 2.1, we can find infinitely many $p, q \in \mathcal{O}$, $q \neq 0$, such that

$$\|\iota_S(q) \cdot \mathbf{x} + \iota_S(p)\| \leq CQ^{-1} \leq C \|\iota_S(q)\|^{-1}.$$

\square

Note that $\mathbf{x} \in \mathbf{BA}_K$ if and only if the constant C in (2.2) cannot be replaced by an arbitrarily small constant. In other words, elements of \mathbf{BA}_K are the witnesses of the optimality of Theorem 2.3.

Remark 2.4.

(i) When $\#S = 1$, $\|\iota_S(q)\|$ can be factored out from the left hand side of (2.2):

$$\|\iota_S(q) \cdot \mathbf{x} + \iota_S(p)\| = \left\| \iota_S(q) \cdot \left(\mathbf{x} + \iota_S\left(\frac{p}{q}\right) \right) \right\| = \|\iota_S(q)\| \cdot \left\| \mathbf{x} + \iota_S\left(\frac{p}{q}\right) \right\|,$$

and that implies (2.2) is equivalent to:

$$\left\| \mathbf{x} + \iota_S\left(\frac{p}{q}\right) \right\| \leq C \|\iota_S(q)\|^{-2}. \quad (2.3)$$

In particular for this case, $\mathbf{x} \in \mathbf{BA}_K$ if and only if there exists $c > 0$ such that for all $p \in \mathcal{O}$, $q \in \mathcal{O} \setminus \{0\}$,

$$\left\| \mathbf{x} + \iota_S\left(\frac{p}{q}\right) \right\| > c \|\iota_S(q)\|^{-2}. \quad (2.4)$$

So for $K = \mathbb{Q}$, (2.4) is exactly (1.1); and for K as in Theorem 1.1. this is the definition used in [ESK10].

(ii) In [Hat07], (2.4) is used to define badly approximable numbers when K is real quadratic or totally complex quartic. In fact, it can be shown that in this case (2.4) is equivalent to (1.3).

2.2. Dani Correspondence and the height function. By abusing notation, we let the diagonal embedding $\iota_S : K \rightarrow K_S$ be extended to matrices $\iota_S : M_{m,n}(K) \rightarrow M_{m,n}(K_S)$ by

$$(\iota_S(A))_{i,j} = \iota_S(A_{i,j}) \quad \text{for } A = (A_{i,j}) \in M_{m,n}(K).$$

We also extend the sup norm in K_S to K_S^2 :

$$\|\vec{z}\| = \max\{\|\mathbf{z}_1\|, \|\mathbf{z}_2\|\} \quad \text{where } \vec{z} = \begin{pmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{pmatrix} \in K_S^2.$$

Also note that a K_S -vector $\vec{z} \in K_S^2$ can be viewed in two ways:

$$\vec{z} = \begin{pmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{pmatrix} \in K_S^2 \quad \text{or} \quad \vec{z} = (\vec{z}_v)_{v \in S} \in \prod_{v \in S} K_v^2.$$

By abusing notation again, we also use $\|\cdot\|$ for the sup norm in K_v^2 . In particular,

$$\|\vec{z}\| = \max\{\|\mathbf{z}_1\|, \|\mathbf{z}_2\|\} = \max_{v \in S} \|\vec{z}_v\|.$$

We let K act naturally on K_S^2 by:

$$a\vec{z} = \begin{pmatrix} \iota_S(a) \cdot \mathbf{z}_1 \\ \iota_S(a) \cdot \mathbf{z}_2 \end{pmatrix}, \quad \text{for } a \in K, \vec{z} \in K_S^2. \quad (2.5)$$

Let $G = \mathrm{SL}_2(K_S) \cong \prod_{v \in S} \mathrm{SL}_2(K_v)$, then $\Gamma = \iota_S(\mathrm{SL}_2(\mathcal{O}))$ is a lattice in G , and we denote the homogeneous space G/Γ by X_K . The isomorphism $K_S \cong \mathbb{R}^d$ induces an embedding $\mathrm{SL}_2(K_S) \hookrightarrow \mathrm{SL}_{2d}(\mathbb{R})$. Hence, X_K can be identified with a proper subset of the space $\mathrm{SL}_{2d}(\mathbb{R})/\mathrm{SL}_{2d}(\mathbb{Z})$ of unimodular lattices in \mathbb{R}^{2d} . Via the map: $g\Gamma \mapsto \iota_S(g)\mathcal{O}^2$, an element $\Lambda \in X_K$ can be identified with a discrete free rank 2 \mathcal{O} -module of K_S^2 having a basis $\{\vec{x}, \vec{y}\}$ such that for every $v \in S$, $\{\vec{x}_v, \vec{y}_v\}$ forms a parallelepiped of area 1 in K_v^2 (see Section 2 of [EGL13] for more details).

Following the ideas of Dani, the space X_K was used in [EGL13] to give an alternative description of badly approximable S -numbers. Let us associate each $\mathbf{x} \in K_S$ with the lattice $\Lambda_{\mathbf{x}} = u_{\mathbf{x}}\Gamma \in X_K$, where $u_{\mathbf{x}} \in G$ is defined by:

$$(u_{\mathbf{x}})_v = \begin{pmatrix} 1 & x_v \\ & 1 \end{pmatrix}. \quad (2.6)$$

Also, for each $t \in \mathbb{R}$, we define $g_t \in G$ by:

$$(g_t)_v = \begin{pmatrix} e^t & \\ & e^{-t} \end{pmatrix}. \quad (2.7)$$

It is shown in [EGL13, Proposition 3.1] that

$$\mathbf{x} \in \mathbf{BA}_K \iff \text{the orbit } \{g_t\Lambda_{\mathbf{x}} : t \geq 0\} \text{ is bounded in } X_K. \quad (2.8)$$

In K_S^2 , there is a height function more suitable for our needs than $\|\cdot\|$, which can be thought as a natural extension of the field norm in K . For a vector $\vec{z} = \begin{pmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{pmatrix} \in K_S^2$, we define the *height* of \vec{z} to be:

$$H(\vec{z}) = H\left(\begin{pmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{pmatrix}\right) := \prod_{v \in S} \max\{|\mathbf{z}_1|_v, |\mathbf{z}_2|_v\}^{d_v} = \prod_{v \in S} \|\vec{z}_v\|^{d_v}. \quad (2.9)$$

We also use the same notation $H(\cdot)$ to define the height on K_S similar to (2.9):

$$H(\mathbf{x}) := \prod_{v \in S} |x_v|^{d_v} \quad \text{for } \mathbf{x} \in K_S.$$

In particular, if $a \in K$ then

$$|N(a)| = H(\iota_S(a)).$$

The following lemma gives an important property of the height function under the action of the group of units \mathcal{O}^\times :

Lemma 2.5 ([EGL13, Lemma 2.4]). *There exists a constant $C \geq 1$ depending only on K such that if $H(\vec{z}) \neq 0$ then there exists a unit $\xi \in \mathcal{O}^\times$ satisfying:*

$$C^{-1} H(\vec{z})^{\frac{1}{d}} \leq \|\iota_v(\xi) \vec{z}_v\| \leq C H(\vec{z})^{\frac{1}{d}} \quad \text{for all } v \in S. \quad (2.10)$$

Note that just as in [KT03, Lemma 5.10], this relation holds for higher dimensions and a more general S with a suitable extension of the height function.

This height function provides a way to measure the size of a lattice in K_S^2 . Namely, let us define

$$\delta_H(\Lambda) = \min\{H(\vec{z}) : \vec{z} \in \Lambda \setminus \{0\}\}. \quad (2.11)$$

Using the above lemma, it is shown in [EGL13, Proposition 2.5] that a subset $A \subseteq X_K$ is relatively compact if and only if $\inf\{\delta_H(\Lambda) : \Lambda \in A\} > 0$. Combining it with (2.8), we arrive at

Proposition 2.6 ([EGL13]).

$$\mathbf{x} \in \mathbf{BA}_K \iff \inf\{\delta_H(g_t \Lambda_{\mathbf{x}}) : t \geq 0\} > 0.$$

Remark 2.7. Applying Moore's Ergodic Theorem and arguing similar to [Dan85, Section 2] or [KM99, Theorem 8.7], it can be deduced from Proposition 2.6 that \mathbf{BA}_K has Lebesgue measure zero.

3. \mathcal{H} -DIFFUSE SETS

Before discussing the modification of Schmidt's game needed for our purposes, in this section we survey the notion of *diffuseness* introduced in [BFK⁺12] in order to describe sets which can serve as nice playgrounds for those games.

Let (X, dist) be a complete metric space. Each pair $B = (x, \rho)$, where $x \in X$ and $\rho > 0$, is called a *formal ball in X* , and we denote corresponding closed ball in X by:

$$\mathbf{B}(B) = \mathbf{B}(x, \rho) := \{y \in X : \text{dist}(x, y) \leq \rho\}.$$

More generally, if \mathcal{L} is a subset of X and $\rho > 0$, we denote by $\mathcal{L}^{(\rho)}$ the ρ -neighborhood of \mathcal{L} :

$$\mathcal{L}^{(\rho)} := \{x \in X : \text{dist}(x, \mathcal{L}) \leq \rho\}.$$

The projection maps from $X \times \mathbb{R}_{>0}$ to the first and second factors are denoted by c and r respectively:

$$c(x, \rho) = x \text{ and } r(x, \rho) = \rho \quad \text{for } (x, \rho) \in X \times \mathbb{R}_{>0}.$$

We define a partial order \preceq on the set $X \times \mathbb{R}_{>0}$ of formal balls:

$$B_1 \preceq B_2 \iff \text{dist}(c(B_1), c(B_2)) \leq r(B_2) - r(B_1).$$

In particular, $B_1 \preceq B_2$ implies $\mathbf{B}(B_1) \subseteq \mathbf{B}(B_2)$, but the converse might not hold in general. Notice that if X is a real Banach space, then $\mathbf{B}(B)$ is uniquely determined by B , but in general, there might exist $B_1 \neq B_2$ such that $\mathbf{B}(B_1) = \mathbf{B}(B_2)$. For instance, we can easily find such B 's when X is the Cantor set with the induced metric from \mathbb{R} .

Now let \mathcal{H} be a non-empty collection of closed subsets of X , and pick $0 < \beta < 1$. Following [BFK⁺12], we say that a subset $Y \subseteq X$ is (\mathcal{H}, β) -diffuse if for every $x \in Y$, there exists $\rho_x > 0$ such that for any $(y, \rho) \in Y \times \mathbb{R}_{>0}$ with $(y, \rho) \preceq (x, \rho_x)$, and for any $\mathcal{L} \in \mathcal{H}$:

$$Y \cap \left(\mathbf{B}(y, \rho) \setminus \mathcal{L}^{(\beta\rho)} \right) \neq \emptyset. \quad (3.1)$$

Note that (\mathcal{H}, β) -diffuseness implies (\mathcal{H}, β') -diffuseness for any $0 < \beta' \leq \beta$. Y is said to be \mathcal{H} -diffuse if it is (\mathcal{H}, β) -diffuse for some $0 < \beta < 1$.

Remark 3.1.

- (i) Our definition of diffuse sets is slightly weaker than the definition used in [BFK⁺12], in that they required ρ_x to be uniformly bounded below for all $x \in Y$.
- (ii) The constant ρ_x at first seems to depend also on β , but notice that if ρ_x works of β , it also works for any $0 < \beta' \leq \beta$.

In the set-up of [BFK⁺12] $X = \mathbb{R}^d$ with the Euclidean metric, and the collection \mathcal{H} consisted of all k -dimensional affine subspaces of \mathbb{R}^d , and the corresponding property was called *k-dimensional diffuseness*. Many examples of k -dimensionally diffuse sets were exhibited there. For example, it is clear that any m -dimensional smooth submanifold of \mathbb{R}^d is k -dimensionally diffuse whenever $m > k$. Also if μ is an absolutely decaying measure on \mathbb{R}^d (see [KLW04, PV05] for definition) then, as shown in [BFK⁺12, Proposition 5.1] $\text{supp } \mu$ is k -dimensionally diffuse for all $1 \leq k < d$.

The following two examples of diffuse sets will be used in the proofs of Corollaries 1.4 and 1.6 in the next section:

Example 3.2. For an arbitrary metric space X , let $\mathcal{H} = \{\{x\} : x \in X\}$ and let $Y = \text{supp } \mu$ be the support of an Ahlfors regular measure μ on X (see (1.5) for the definition); then Y is \mathcal{H} -diffuse.

To see this, for every $0 < \beta < \left(\frac{a}{b}\right)^{1/\delta}$, $(y, \rho) \in X \times \mathbb{R}_{>0}$ with $\rho \leq r_0$, and $x \in X$, write

$$\mu(\mathbf{B}(y, \rho) \setminus \mathbf{B}(x, \beta\rho)) \geq a\rho^\delta - b(\beta\rho)^\delta > 0.$$

In particular, $Y \cap (\mathbf{B}(y, \rho) \setminus \mathbf{B}(x, \beta\rho)) \neq \emptyset$; that is, (3.1) holds.

Proposition 3.3. Let \mathcal{S} be a closed set of linear subspaces in \mathbb{R}^n , and let

$$\mathcal{H} = \{\mathcal{L} + \vec{x} : \mathcal{L} \in \mathcal{S}, \vec{x} \in \mathbb{R}^n\}.$$

If $\phi : [0, 1] \rightarrow \mathbb{R}^n$ is an injective C^1 map such that the curve $\phi([0, 1])$ is not tangent to any affine subspace in \mathcal{H} , then $\phi([0, 1])$ is \mathcal{H} -diffuse.

Proof. For every $\mathcal{L} \in \mathcal{S}$, let $\pi_{\mathcal{L}^\perp} : \mathbb{R}^n \rightarrow \mathcal{L}^\perp$ be the projection onto its orthogonal complement. Let

$$a = \min_{\substack{0 \leq t \leq 1 \\ \mathcal{L} \in \mathcal{S}}} \frac{\|(\pi_{\mathcal{L}^\perp} \circ \phi)'(t)\|}{\|\phi'(t)\|}, \quad b = \max_{0 \leq t \leq 1} \|\phi'(t)\| \quad \text{and} \quad c = \min_{0 \leq t \leq 1} \|\phi'(t)\|.$$

Note that $a > 0$ by the non-tangency and compactness of \mathcal{S} . In particular, for every $0 \leq t \leq 1$ and $\mathcal{L} \in \mathcal{S}$:

$$\|(\pi_{\mathcal{L}^\perp} \circ \phi)'(t)\| \geq ac.$$

For every $\vec{x} \in \phi([0, 1])$, we denote $t_{\vec{x}} = \phi^{-1}(\vec{x})$. For an arbitrary $\vec{x} \in \phi([0, 1])$ (the proof is similar at the other endpoint), we let $\rho_{\vec{x}} > 0$ be sufficiently small such that

$$\partial \mathbf{B}(\vec{x}, \rho_{\vec{x}}) \cap \phi((t_{\vec{x}}, 1)) \neq \emptyset;$$

and for $\vec{y}, \vec{z} \in \mathbf{B}(\vec{x}, \rho_{\vec{x}})$ and for every $\mathcal{L} \in \mathcal{S}$:

$$\frac{\|\pi_{\mathcal{L}^\perp}(\vec{y} - \vec{z})\|}{|t_{\vec{y}} - t_{\vec{z}}|} \geq \frac{ac}{2}.$$

Let $0 < \beta < \frac{ac}{4b\sqrt{n}}$, and $(\vec{y}, \rho) \in \phi([0, 1]) \times \mathbb{R}_{>0}$ be arbitrary with $(\vec{y}, \rho) \preceq (\vec{x}, \rho_{\vec{x}})$. Let

$$t_0 = \min \phi^{-1}(\partial \mathbf{B}(\vec{y}, \rho) \cap [t_{\vec{y}}, 1]) = \min \{t > t_{\vec{y}} : \|\phi(t) - \vec{y}\| = \rho\}.$$

Let $\mathcal{L} + \vec{z} \in \mathcal{H}$ be arbitrary. For any $0 \leq t_1 < t_2 \leq 1$ such that $\phi(t_1), \phi(t_2) \in \mathbf{B}(\vec{y}, \rho) \cap (\mathcal{L} + \vec{z})^{(\beta\rho)}$, the arclength of $\phi([t_1, t_2])$ is:

$$\begin{aligned} l(\phi([t_1, t_2])) &= \int_{t_1}^{t_2} \sqrt{|\phi'_1(t)|^2 + \dots + |\phi'_n(t)|^2} dt \\ &\leq |t_2 - t_1| \cdot b\sqrt{n} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{2}{ac} \|\pi_{\mathcal{L}^\perp}(\phi(t_2) - \phi(t_1))\| \cdot b\sqrt{n} \\
&\leq \frac{2b}{ac} \cdot 2\beta\rho \cdot \sqrt{n} < \rho \leq l(\phi([t_{\vec{y}}, t_0])).
\end{aligned}$$

In particular, for any $\mathcal{L} + \vec{z} \in \mathcal{H}$:

$$\phi([0, 1]) \cap \left(\mathbf{B}(\vec{y}, \rho) \setminus (\mathcal{L} + \vec{z})^{(\beta\rho)} \right) \neq \emptyset.$$

This shows that $\phi([0, 1])$ is \mathcal{H} -diffuse. \square

We close the section with the following useful property of diffuse sets:

Lemma 3.4. *If Y is (\mathcal{H}, β) -diffuse, then for any $0 < \beta' \leq \frac{\beta}{2 + \beta}$, $x \in Y$, $(y, \rho) \in Y \times \mathbb{R}_{>0}$ with $(y, \rho) \preceq (x, \rho_x)$, and any $\mathcal{L} \in \mathcal{H}$, there exists $z \in Y \cap \mathbf{B}(y, \rho)$ with*

$$(z, \beta'\rho) \preceq (y, \rho) \quad \text{and} \quad \text{dist}(z, \mathcal{L}) > 2\beta'\rho. \quad (3.2)$$

Proof. The proof is identical to the proof of [BFK⁺12, Lemma 4.3] which stated the same result in the k -dimensional diffuseness set-up. It is sufficient to prove the lemma for $\beta' = \frac{\beta}{2 + \beta} < \beta$. Since $(y, (1 - \beta')\rho) \preceq (y, \rho) \preceq (x, \rho_x)$, there exists

$$z \in Y \cap \left(\mathbf{B}(y, (1 - \beta')\rho) \setminus \mathcal{L}^{(\beta(1 - \beta')\rho)} \right) = Y \cap \left(\mathbf{B}(y, (1 - \beta')\rho) \setminus \mathcal{L}^{(2\beta'\rho)} \right).$$

Thus (3.2) holds. \square

4. SCHMIDT GAME AND ITS VARIANTS

4.1. Schmidt's (α, β) -game. In [Sch66], W. Schmidt introduced an infinite game between two players, called Alice and Bob, which has been shown to be a powerful tool in Diophantine approximation. The setup of Schmidt's game requires the followings:

- (1) The playground is a complete metric space (X, dist) .
- (2) Two real numbers α, β , with $0 < \alpha, \beta < 1$, are parameters associated with Alice and Bob respectively.
- (3) A subset $W \subseteq X$ is Alice's *target set*.

The game proceeds by Alice and Bob alternatively picking $B_1, A_1, B_2, A_2, \dots$ with $A_n, B_n \in X \times \mathbb{R}_{>0}$ satisfying:

- (S1) B_n 's are Bob's choices, A_n 's are Alice choices.
- (S2) $r(A_n) = \alpha r(B_n)$ and $A_n \preceq B_n$ for all $n \geq 1$.
- (S3) $r(B_n) = \beta r(A_{n-1})$ and $B_{n+1} \preceq A_n$ for all $n \geq 2$.

In particular, the result of a play forms a nested sequence of closed balls:

$$\mathbf{B}(B_1) \supseteq \mathbf{B}(A_1) \supseteq \mathbf{B}(B_2) \supseteq \mathbf{B}(A_2) \supseteq \dots$$

Since X is complete, and the radii $\lim_{n \rightarrow \infty} r(A_n) = \lim_{n \rightarrow \infty} r(B_n) = 0$, their intersection is a point, denoted by x_∞ :

$$\bigcap_{n=1}^{\infty} \mathbf{B}(A_n) = \bigcap_{n=1}^{\infty} \mathbf{B}(B_n) = \{x_\infty\}.$$

If Alice has a strategy that guarantees $x_\infty \in W$ regardless of what Bob does, then we say that W is (α, β) -winning. If W is (α, β) -winning for every $0 < \beta < 1$, then it is called α -winning. And finally, W is called *winning* if it is α -winning for some $0 < \alpha < 1$.

Schmidt showed that winning sets have remarkable properties:

Lemma 4.1 ([Sch66]).

- (i) If $X = \mathbb{R}^n$ then winning sets are thick.
- (ii) A countable intersection of α -winning sets is again α -winning.
- (iii) If W is α -winning and $\phi : X \rightarrow X$ is bi-Lipschitz, then $\phi(W)$ is α' -winning, with α' depends on α and the bi-Lipschitz constant of ϕ .

4.2. \mathcal{H} -absolute game on \mathcal{H} -diffuse sets. Generalizing the ideas of the absolute game of McMullen [McM10] and the k -dimensional absolute game of Broderick et al. [BFK⁺12], Fishman, Simmons and Urbanski [FSU13] introduced the \mathcal{H} -absolute game which we will describe as follows.

Let (X, dist) be a complete metric space, let \mathcal{H} be a non-empty collection of closed subsets of X , and pick $0 < \beta < 1$. For a non-empty closed subset $Y \subseteq X$, Alice and Bob play the \mathcal{H} -absolute game on Y by alternatively choosing an infinite sequence $B_1, A_1, B_2, A_2, \dots$ satisfying the rules below:

- (H1) Bob chooses $B_n = (c_n, r_n) \in Y \times \mathbb{R}_{>0}$.
- (H2) Alice chooses $A_n = (\mathcal{L}_n, \rho_n) \in \mathcal{H} \times \mathbb{R}_{>0}$ with $0 < \rho_n \leq \beta r_n$.
- (H3) $\beta r_n \leq r_{n+1} \leq r_n$ and $B_{n+1} \subseteq B_n$.
- (H4) $\text{dist}(c_{n+1}, \mathcal{L}_n) > \rho_n + r_{n+1}$.
- (H5) If at some point of the game, Bob has no choices $B_{n+1} \in X \times \mathbb{R}_{>0}$ satisfying (H3) and (H4), then Bob wins, and the game is terminated.

In particular, conditions (H3) and (H4) imply that $\mathbf{B}(B_{n+1}) \subseteq \mathbf{B}(B_n) \setminus \mathcal{L}_n^{(\rho_n)}$. Hence, we can think of Alice's move as deleting a neighborhood of a closed subset \mathcal{L} in \mathcal{H} .

Remark 4.2. As a convention, the only neighborhood of the empty set is the empty set itself, and the distance from any point of X to the empty set is infinite. Hence, the empty set, if in \mathcal{H} , can be considered as a 'dummy move' for Alice.

A set $W \subseteq X$ is said to be (\mathcal{H}, β) -absolute winning on Y if Alice has a strategy to ensure that at every stage of the game, Bob always has at least one choice, and:

$$W \cap Y \cap \bigcap_{n=1}^{\infty} \mathbf{B}(B_n) \neq \emptyset$$

regardless of Bob's strategy. We will say that W is \mathcal{H} -absolute winning on Y if there exists $\beta_0 > 0$ such that W is (\mathcal{H}, β) -absolute winning on Y for all $0 < \beta < \beta_0$.

Remark 4.3. When $Y = X$ we will drop 'on Y ' and simply say that W is (\mathcal{H}, β) -absolute winning and \mathcal{H} -absolute winning accordingly.

Remark 4.4. It is clear from the rules of the game that for any collection \mathcal{H} , if $W \subseteq X$ is \mathcal{H} -absolute winning on Y , then $W \cap Y$ is dense in Y with the subspace topology, since that implies $W \cap Y \cap \mathbf{B}(B_1) \neq \emptyset$ for arbitrary B_1 .

Example 4.5.

- (i) When $\mathcal{H} = \{\{x\} : x \in X\}$ is the set of singletons in X , the \mathcal{H} -absolute game is the absolute game considered by McMullen [McM10].
- (ii) When $X = \mathbb{R}^d$ and \mathcal{H} is the collection of k -dimensional affine subspaces in \mathbb{R}^d , we get the k -dimensional absolute game of Broderick et al [BFK⁺12]. See also [BFS13, NS14, KW15, AGK15] for other appearances of games of this type.

The degenerate case when Alice's moves leave Bob without any legitimate choice at some point of the game can be avoided when $X = Y = \mathbb{R}^n$ by restricting $0 < \beta < 1/3$ as in [McM10, BFK⁺12]. For the k -dimensional absolute game playing on a subset Y of \mathbb{R}^n , Broderick et al. [BFK⁺12] showed that the k -dimensional diffuseness of Y is a sufficient condition for the game to last infinitely. More generally, Lemma 3.4, essentially taken from [BFK⁺12], makes sure that Bob always has legitimate moves in the \mathcal{H} -absolute game played on Y if Y is \mathcal{H} -diffuse, allowing us to ignore condition (H5):

Remark 4.6. By Lemma 3.4, to show that W is \mathcal{H} -absolute winning on an \mathcal{H} -diffuse set Y , it suffices to assume that $r(B_n) \rightarrow 0$. So by rearranging the indices, it suffices to assume that $\rho_{c(B_1)} = r(B_1)$. So if $B_1 = (x, \rho) \in Y \times \mathbb{R}_{>0}$ be Bob's arbitrary first move, then $r(B_N) < \rho_x$ for all N sufficiently large. By rearranging the indices, we can assume that the $\rho_{c(B_1)} = \rho_x = r(B_1)$.

We now list various properties of \mathcal{H} -absolute winning sets. To make the exposition more streamlined, their proofs are postponed until Appendix B.

The next lemma shows that a set \mathcal{H} -absolute winning on an \mathcal{H} -diffuse set will be absolute winning for all reasonable β 's:

Lemma 4.7. *If W is \mathcal{H} -absolute winning on Y and Y is (\mathcal{H}, β) -diffuse, then W is (\mathcal{H}, β') -absolute winning on Y for all $0 < \beta' \leq \frac{\beta}{2 + \beta}$.*

Proof. See §B.1. □

Using the above lemma, we can derive that \mathcal{H} -absolute winning on \mathcal{H} -diffuse set implies winning in Schmidt's sense:

Proposition 4.8. *If W is \mathcal{H} -absolute winning on Y , and Y is (\mathcal{H}, β) -diffuse, then $W \cap Y$ is $\frac{\beta}{2 + \beta}$ -winning when we play Schmidt game on Y equipped with the induced metric.*

Proof. See §B.2. □

Following Schmidt's idea, we will show that \mathcal{H} -absolute winning sets possess countable intersection property:

Proposition 4.9. *Let Y be an (\mathcal{H}, β) -diffuse set. Then a countable intersection of sets \mathcal{H} -absolute winning on Y is also \mathcal{H} -absolute winning on Y .*

Proof. See §B.3. □

An easy consequence of the countable intersection property is that the \mathcal{H} -absolute winning property remains if we discard a countable number of removable points from the target set:

Proposition 4.10. *Let Y be an \mathcal{H} -diffuse set, W be an \mathcal{H} -absolute winning on Y , and $Z \subseteq W \cap \bigcup_{\mathcal{L} \in \mathcal{H}} (\mathcal{L} \cap Y)$ is countable. Then $W \setminus Z$ is \mathcal{H} -absolute winning on Y .*

Proof. See §B.4. □

Probably the most importance property of diffuse sets is the inheritance property whose proof is verbatim to the proof of [BFK⁺12, Proposition 4.9] for the k -dimensional absolute game:

Proposition 4.11 ([BFK⁺12, Proposition 4.9]). *If $Y \subseteq Z$ are both \mathcal{H} -diffuse and W is \mathcal{H} -absolute winning on Z , then W is also \mathcal{H} -absolute winning on Y .*

In the proof of Theorem 1.3, we will use a version of the \mathcal{H} -absolute game, in which Alice is allowed to choose N closed subsets $\mathcal{L}_1, \dots, \mathcal{L}_N$ in \mathcal{H} in each move:

$$A_n = \left(\bigcup_{i=1}^N \mathcal{L}_i, \rho \right)$$

for a fixed $N \geq 1$. Let

$$\mathcal{H}^{*N} = \left\{ \bigcup_{i=1}^N \mathcal{L}_i : \mathcal{L}_i \in \mathcal{H} \text{ for } 1 \leq i \leq N \right\}.$$

We will show that this change in the rule won't affect the class of \mathcal{H} -absolute winning sets:

Proposition 4.12. *Assume that Y is \mathcal{H} -diffuse, and let $W \subseteq X$. Then the followings are equivalent:*

- (i) W is \mathcal{H} -absolute winning on Y .
- (ii) W is \mathcal{H}^{*N} -absolute winning on Y .

Proof. See §B.5. □

Remark 4.13. Technically speaking, a strategy for Alice in the game described above will have to take all the previous moves in consideration. Nevertheless, it follows from [Sch66, Theorem 7] that for a general topological infinite game of two players, including both games described in §4.1 and §4.2, if Alice is winning then there exists a winning strategy for Alice that only takes Bob's immediately preceding move into account. Such strategy is called a *positional winning strategy*, and for our interest in the case of the \mathcal{H} -absolute game, it can be defined as a function $\sigma : Y \times \mathbb{R}_{>0} \rightarrow \mathcal{H} \times \mathbb{R}_{>0}$ satisfying:

- (i) For any $B_n \in Y \times \mathbb{R}_{>0}$, $A_n = \sigma(B_n)$ satisfies (H2),
- (ii) For any sequence $B_1, A_1 = \sigma(B_1), B_2, A_2 = \sigma(B_2), \dots$ satisfying (H1)–(H4), Bob always has available choices at every stage of the game, and the intersection:

$$W \cap Y \bigcap_{n=1}^{\infty} \mathbf{B}(B_n) \neq \emptyset.$$

We end this section by proving Corollaries 1.6 and 1.4 assuming Theorem 1.3.

Proof of Corollary 1.4 and Theorem 1.2. Recall that for every $t \in [0, 1]$ we have defined

$$T(t) = \{v \in S : \phi'_v(t) \neq 0\}.$$

Let

$$D := \left\{ t \in [0, 1] : \#T(t)_{\mathbb{R}} + 2(\#T(t)_{\mathbb{C}}) \leq \frac{d}{2} \right\}.$$

First, we will assume that $D = \emptyset$ and ϕ is injective. Then by Proposition 3.3, $\phi([0, 1])$ is \mathcal{H}_K -diffuse. So by Theorem 1.3 and Proposition 4.11, \mathbf{BA}_K is \mathcal{H}_K -absolute winning on $\phi([0, 1])$.

Let a, b, c be defined as in the proof of Proposition 3.3, and let $0 < \beta < \frac{1}{3}$ be sufficiently small such that $\phi([0, 1])$ is $\left(\mathcal{H}_K, \frac{2\beta'}{1-\beta'}\right)$ -diffuse for $\beta' = \beta \cdot \min\left\{1, \frac{ac}{4b\sqrt{d}}\right\}$. In particular, \mathbf{BA}_K is (\mathcal{H}_K, β') -absolute winning on $\phi([0, 1])$. Let $\sigma : \phi([0, 1]) \times \mathbb{R}_{>0} \rightarrow \mathcal{H}_K \times \mathbb{R}_{>0}$ be a positional winning strategy (see Remark 4.13 above). Consider the (\mathcal{H}, β) -absolute game on $[0, 1]$ with $\mathcal{H} = \{\{t\} : 0 \leq t \leq 1\}$, and let $B_n = (t_n, r_n) \in [0, 1] \times \mathbb{R}_{>0}$ be Bob's arbitrary move. It suffices to assume that $r_nb\sqrt{d} < \rho_{\phi(t_n)}$.

For every $t_n \leq t \leq t_n + r_n$, the arc length of $\phi([t_n, t])$ is:

$$l(\phi([t_n, t])) \leq (t - t_n)b\sqrt{d} \leq r_nb\sqrt{d}.$$

That implies:

$$\phi(\mathbf{B}(B_n)) \subseteq \mathbf{B}\left(\phi(t_n), r_nb\sqrt{d}\right).$$

Let

$$B'_n = \left(\phi(t_n), r_nb\sqrt{d}\right) \quad \text{and} \quad A'_n = \sigma(B'_n) = (\mathcal{L}_n, \rho'_n).$$

By the Mean Value Theorem, \mathcal{L}_n intersects $\phi([0, 1])$ at at most 1 point. If

$$\phi^{-1}\left(\mathcal{L}_n^{(\rho'_n)} \cap \phi([0, 1])\right) \cap \mathbf{B}(B_n) = \emptyset$$

then Alice can make an arbitrary move A_n in the (\mathcal{H}, β) -absolute game on $[0, 1]$. If $\mathcal{L}_n \cap \mathbf{B}(B_n) \neq \emptyset$, then let $A_n = (\phi^{-1}(\mathcal{L}_n \cap \mathbf{B}(B_n)), \beta r_n)$, otherwise, let $A_n = (\phi^{-1}(\mathbf{x}), \beta r_n)$ for arbitrary $\mathbf{x} \in \mathcal{L}_n^{(\rho'_n)} \cap \mathbf{B}(B'_n) \cap \phi([0, 1])$. So for any $\mathbf{y} \in \mathcal{L}_n^{(\rho'_n)} \cap \mathbf{B}(B'_n) \cap \phi([0, 1])$,

$$|c(A_n) - \phi^{-1}(\mathbf{y})| \leq \frac{2}{ac} \|\pi_{\mathcal{L}^\perp}(\mathbf{y} - \phi(c(A_n)))\| \leq \frac{2}{ac} 2\beta' r_n b \sqrt{d} \leq \beta r_n.$$

It shows that:

$$\phi(\mathbf{B}(A_n)) \supseteq \mathcal{L}_n^{(\rho'_n)} \cap \mathbf{B}(B'_n) \cap \phi([0, 1]).$$

Moreover, since $\beta' \leq \beta$,

$$\|\phi(t_{n+1}) - \phi(t_n)\| \leq (1 - \beta) r_n b \sqrt{d} \leq (1 - \beta') r_n b \sqrt{d}.$$

So the sequence $B'_n, \sigma(B'_n), B'_{n+1}, \sigma(B'_{n+1}), \dots$ satisfies the conditions (H1)–(H5) of the (\mathcal{H}_K, β') -absolute game on $\phi([0, 1])$. Therefore,

$$\phi\left(\bigcap_{n=1}^{\infty} \mathbf{B}(B_n)\right) \subseteq \bigcap_{n=1}^{\infty} \mathbf{B}(B'_n) \cap \phi([0, 1]) \subseteq \mathbf{BA}_K \cap \phi([0, 1]).$$

Hence, $\phi^{-1}(\mathbf{BA}_K)$ is absolute winning on $[0, 1]$.

In the second case when $D = \emptyset$ and ϕ is not injective, by the Constant Rank Theorem, we can cover the interval $[0, 1]$ so that ϕ is injective on each subinterval. Applying the first case, we have that $\phi([0, 1])$ is \mathcal{H}_K -diffuse and $\phi^{-1}(\mathbf{BA}_K)$ is absolute winning.

Finally, when $D \neq \emptyset$, then by the previous case, for every $n \geq 1$:

$$W_n := \phi^{-1}(\mathbf{BA}_K) \cup \bigcup_{t \in D} \left(\left(t - \frac{1}{n}, t + \frac{1}{n} \right) \cap [0, 1] \right)$$

is absolute winning on $[0, 1]$.

Since

$$D = \left\{ t \in [0, 1] : \#T(t)_{\mathbb{R}} + 2(\#T(t)_{\mathbb{C}}) \leq \frac{d}{2} \right\} = \bigcup_{\substack{T \subseteq S \\ \#T_{\mathbb{R}} + 2(\#T_{\mathbb{C}}) > \frac{d}{2}}} \bigcap_{v \in T} (\phi'_v)^{-1}(0),$$

D is closed, and hence,

$$\phi^{-1}(\mathbf{BA}_K) \cup D = \bigcap_{n=1}^{\infty} W_n.$$

Applying Proposition 4.9, $\phi^{-1}(\mathbf{BA}_K) \cup D$ is absolute winning on $[0, 1]$. Thus, by Proposition 4.10, $\phi^{-1}(\mathbf{BA}_K)$ is absolute winning on $[0, 1]$. □

Remark 4.14. The corollary still holds with $[0, 1]$ replaced by \mathbb{R} .

Proof of Corollary 1.6 and Theorem 1.1. Let K be an imaginary quadratic field and Y be the support of an Ahlfors regular measure. Consider McMullen's absolute game on \mathbb{C} where $\mathcal{H} = \{\{x\} : x \in \mathbb{C}\}$. Since \mathbf{BA}_K is absolute winning by Corollary 1.5 and Y is \mathcal{H} -diffuse by Example 3.2, it follows from Proposition 4.11 that \mathbf{BA}_K is absolute winning on Y . Thus, $\mathbf{BA}_K \cap Y$ is winning for the Schmidt game playing on Y by Proposition 4.8. □

5. PROOF OF THEOREM 1.3

One of the key ingredients to show the winning property has traditionally been the Simplex Lemma, see [BFK⁺12] and references therein. The next statement (Lemma 5.1) can be thought of as a number field analogue of the Simplex Lemma. Together with an estimate of the growth of the height function with respect to the flow $g_t \Lambda_{\mathbf{x}}$ as \mathbf{x} varies in Lemma 5.2, it implies Lemma 5.3 which is a more refined version of the Simplex Lemma.

Lemma 5.1 ([EGL13, Lemma 4.1]). *Let $u \in G$, and $\vec{z} = u \iota_S \begin{pmatrix} p \\ q \end{pmatrix}, \vec{z}' = u \iota_S \begin{pmatrix} p' \\ q' \end{pmatrix}$, where $p, p' \in \mathcal{O}$, $q, q' \in \mathcal{O} \setminus \{0\}$. If $H(\vec{z}) H(\vec{z}') < 2^{-d}$, then $\frac{p}{q} = \frac{p'}{q'}$.*

For completeness, we will provide a proof of Lemma 5.1 here, which is the same as the proof in [EGL13] with a minor correction on the constant.

Proof. Let $P = \iota_S \begin{pmatrix} p & p' \\ q & q' \end{pmatrix} \in M_{2,2}(K_S)$, then the height of the determinant of uP is:

$$\begin{aligned} H(\det(uP)) &= H(\det(P)) && (\text{since } u \in \text{SL}_2(K_S)) \\ &= H(\iota_S(pq' - p'q)) \\ &= |N(pq' - p'q)| && (\text{since } pq' - p'q \in \mathcal{O}) \end{aligned}$$

On the other hand,

$$\begin{aligned} H(\det(uP)) &= H\left(\det \begin{pmatrix} \mathbf{z}_1 & \mathbf{z}'_1 \\ \mathbf{z}_2 & \mathbf{z}'_2 \end{pmatrix}\right) \\ &= H(\mathbf{z}_1 \mathbf{z}'_2 - \mathbf{z}'_1 \mathbf{z}_2) \\ &= \prod_{v \in S} |(\mathbf{z}_1)_v (\mathbf{z}'_2)_v - (\mathbf{z}'_1)_v (\mathbf{z}_2)_v|^{d_v} \\ &\leq \prod_{v \in S} (2 \max\{|(\mathbf{z}_1)_v|, |(\mathbf{z}_2)_v|\} \cdot \max\{|(\mathbf{z}'_1)_v|, |(\mathbf{z}'_2)_v|\})^{d_v} \\ &= 2^d H(\vec{z}) H(\vec{z}') < 1. \end{aligned}$$

Thus, $|N(pq' - p'q)| = 0$ and $pq' = p'q$, since $|N(pq' - p'q)| \in \mathbb{N}$. □

Lemma 5.2. *Let $\mathbf{x}, \mathbf{y} \in K_S$ such that $\|\mathbf{x} - \mathbf{y}\| \leq \rho$. Then for any $p \in \mathcal{O}$, $q \in \mathcal{O} \setminus \{0\}$ and for any $t \geq 0$,*

$$(1 + e^{2t} \rho)^{-d} H\left(g_t u_{\mathbf{x}} \iota_S \begin{pmatrix} p \\ q \end{pmatrix}\right) \leq H\left(g_t u_{\mathbf{y}} \iota_S \begin{pmatrix} p \\ q \end{pmatrix}\right) \leq (1 + e^{2t} \rho)^d H\left(g_t u_{\mathbf{x}} \iota_S \begin{pmatrix} p \\ q \end{pmatrix}\right).$$

Proof.

$$\begin{aligned} \frac{H\left(g_t u_{\mathbf{y}} \iota_S \begin{pmatrix} p \\ q \end{pmatrix}\right)}{H\left(g_t u_{\mathbf{x}} \iota_S \begin{pmatrix} p \\ q \end{pmatrix}\right)} &= \prod_{v \in S} \frac{\max\{e^{-t} |\iota_v(q)|, e^t |\iota_v(q) y_v + \iota_v(p)|\}^{d_v}}{\max\{e^{-t} |\iota_v(q)|, e^t |\iota_v(q) x_v + \iota_v(p)|\}^{d_v}} \\ &= \prod_{v \in S} \frac{\max\left\{e^{-t}, e^t \left|y_v + \iota_v\left(\frac{p}{q}\right)\right|\right\}^{d_v}}{\max\left\{e^{-t}, e^t \left|x_v + \iota_v\left(\frac{p}{q}\right)\right|\right\}^{d_v}} \end{aligned}$$

$$\begin{aligned}
& \leq \prod_{v \in S} \frac{\max \left\{ e^{-t}, e^t \left(|y_v - x_v| + \left| x_v + \iota_v \left(\frac{p}{q} \right) \right| \right) \right\}^{d_v}}{\max \left\{ e^{-t}, e^t \left| x_v + \iota_v \left(\frac{p}{q} \right) \right| \right\}^{d_v}} \\
& \leq \prod_{v \in S} \frac{\max \left\{ e^{-t}, e^t \left(\rho + \left| x_v + \iota_v \left(\frac{p}{q} \right) \right| \right) \right\}^{d_v}}{\max \left\{ e^{-t}, e^t \left| x_v + \iota_v \left(\frac{p}{q} \right) \right| \right\}^{d_v}} \\
& \leq \prod_{v \in S} (1 + e^{2t} \rho)^{d_v} \\
& = (1 + e^{2t} \rho)^d.
\end{aligned}$$

The reverse inequality is obtained by symmetry. \square

Lemma 5.3. *Let $\mathbf{B} = \mathbf{B}(\mathbf{y}, \rho)$. Let $t > 0$ and $\varepsilon = 2^{-d}(1 + 2e^{2t}\rho)^{-d}$. If there exists $p \in \mathcal{O}$, $q \in \mathcal{O} \setminus \{0\}$ and $\mathbf{x} \in \mathbf{B}$ such that*

$$\mathbf{H} \left(g_t u_{\mathbf{x}} \iota_S \left(\frac{p}{q} \right) \right) \leq \varepsilon,$$

then for every $p' \in \mathcal{O}$, $q' \in \mathcal{O} \setminus \{0\}$ with $\frac{p'}{q'} \neq \frac{p}{q}$, and for every $\mathbf{x}' \in \mathbf{B}$:

$$\mathbf{H} \left(g_t u_{\mathbf{x}'} \iota_S \left(\frac{p'}{q'} \right) \right) \geq 1.$$

Proof. Assume by contradiction that there exists $\mathbf{x}' \in \mathbf{B}$, $p' \in \mathcal{O}$, $q' \in \mathcal{O} \setminus \{0\}$, $\frac{p}{q} \neq \frac{p'}{q'}$, such that

$$\mathbf{H} \left(g_t u_{\mathbf{x}'} \iota_S \left(\frac{p'}{q'} \right) \right) < 1.$$

Then by Lemma 5.2,

$$\mathbf{H} \left(g_t u_{\mathbf{x}} \iota_S \left(\frac{p'}{q'} \right) \right) \leq \mathbf{H} \left(g_t u_{\mathbf{x}'} \iota_S \left(\frac{p'}{q'} \right) \right) \cdot (1 + e^{2t} \|\mathbf{x} - \mathbf{x}'\|)^d < (1 + 2e^{2t} \rho)^d.$$

That implies

$$\mathbf{H} \left(g_t u_{\mathbf{x}} \iota_S \left(\frac{p}{q} \right) \right) \cdot \mathbf{H} \left(g_t u_{\mathbf{x}} \iota_S \left(\frac{p'}{q'} \right) \right) < \varepsilon \cdot (1 + 2e^{2t} \rho)^d = 2^{-d},$$

contradicting Lemma 5.1. \square

Let $f(t)$ be a function of t . Denote the forward derivative of f at t to be:

$$\frac{d^+}{dt} (f(t)) = \lim_{h \rightarrow 0^+} \frac{f(t+h) - f(t)}{h}, \quad (5.1)$$

then for any $\mathbf{x} \in K_S$, $p \in \mathcal{O}$, $q \in \mathcal{O} \setminus \{0\}$, $\frac{d^+}{dt} \mathbf{H} \left(g_t u_{\mathbf{x}} \iota_S \left(\frac{p}{q} \right) \right)$ exists for every $t \geq 0$. Note

that the two-sided derivative $\frac{d}{dt} \mathbf{H} \left(g_t u_{\mathbf{x}} \iota_S \left(\frac{p}{q} \right) \right)$ will fail to exist at the time t for which $e^{-t} = e^t |\iota_v(q)x_v + \iota_v(p)|$ for some $v \in S$.

Moreover, if we denote

$$T_{\mathbf{x},t} = \left\{ v \in S : \left| x_v + \iota_v \left(\frac{p}{q} \right) \right| < e^{-2t} \right\}, \quad (5.2)$$

then

$$\begin{aligned}
\frac{d^+}{dt} \log H\left(g_t u_{\mathbf{x}} \iota_S\left(\frac{p}{q}\right)\right) &= \frac{d^+}{dt} \log \prod_{v \in S} \max\{e^{-t} |\iota_v(q)|, e^t |\iota_v(q) x_v + \iota_v(p)|\}^{d_v} \\
&= \sum_{v \in S} d_v \frac{d^+}{dt} \log \max\{e^{-t} |\iota_v(q)|, e^t |\iota_v(q) x_v + \iota_v(p)|\} \\
&= \sum_{v \in S} d_v \frac{d^+}{dt} \left(\log \max\left\{e^{-t}, e^t \left|x_v + \iota_v\left(\frac{p}{q}\right)\right|\right\} + \log |\iota_v(q)| \right) \\
&= \sum_{v \in S \setminus T_{\mathbf{x},t}} d_v \frac{d^+}{dt} \log e^t \left|x_v + \iota_v\left(\frac{p}{q}\right)\right| + \sum_{v \in T_{\mathbf{x},t}} d_v \frac{d^+}{dt} \log e^{-t} \\
&= \sum_{v \in S \setminus T_{\mathbf{x},t}} d_v - \sum_{v \in T_{\mathbf{x},t}} d_v.
\end{aligned} \tag{5.3}$$

This gives us a trivial bound of the forward derivatives:

$$\left| \frac{d^+}{dt} \log H\left(g_t u_{\mathbf{x}} \iota_S\left(\frac{p}{q}\right)\right) \right| \leq d. \tag{5.4}$$

Remark 5.4. It also follows from (5.3) that

$$\#(T_{\mathbf{x},t})_{\mathbb{R}} + 2 \cdot \#(T_{\mathbf{x},t})_{\mathbb{C}} \leq \frac{d}{2}$$

if and only if

$$\frac{d^+}{dt} \log H\left(g_t u_{\mathbf{x}} \iota_S\left(\frac{p}{q}\right)\right) \geq 0.$$

From this remark, we can deduce that every subspace of \mathcal{H}_K defined in (1.7) does not contain any badly approximable S -numbers:

Proposition 5.5. *For any $\mathcal{L} \in \mathcal{H}_K$,*

$$\mathcal{L} \cap \mathbf{BA}_K = \emptyset.$$

Proof. Let $\mathbf{y} \in \mathcal{L} = \mathcal{L}(\mathbf{x}, T) \in \mathcal{H}_K$ be arbitrary, and let $p, q \in \mathcal{O}$ such that $\mathbf{x} = \iota_S\left(\frac{p}{q}\right)$. Since for every $v \in T$,

$$\left| \mathbf{y}_v - \iota_v\left(\frac{p}{q}\right) \right| = |\mathbf{y}_v - \mathbf{x}_v| = 0,$$

we have that for every $t \geq 0$,

$$T \subseteq T_{\mathbf{y},t}.$$

In particular,

$$\#(T_{\mathbf{y},t})_{\mathbb{R}} + 2 \cdot \#(T_{\mathbf{y},t})_{\mathbb{C}} > \frac{d}{2}.$$

So by Remark 5.4, for every $t \geq 0$,

$$\frac{d^+}{dt} \log H\left(g_t u_{\mathbf{y}} \iota_S\left(\frac{p}{q}\right)\right) \leq -1.$$

That implies

$$\lim_{t \rightarrow \infty} H\left(g_t u_{\mathbf{y}} \iota_S\left(\frac{p}{q}\right)\right) = 0.$$

Thus, $\mathbf{y} \notin \mathbf{BA}_K$. □

Moreover, the forward derivatives provide us another criterion for an $\mathbf{x} \in K_S$ to be badly approximable, which will be used in the proof of Theorem 1.3:

Lemma 5.6. $\mathbf{x} \in \mathbf{BA}_K$ if and only if there exists $\varepsilon > 0, c > 0$, and a sequence $0 < t_1 < t_2 < \dots$ with $\lim_{n \rightarrow \infty} t_n = \infty$ and $t_{n+1} - t_n \leq c$ such that for every $p, q \in \mathcal{O}, (p, q) \neq (0, 0)$,

$$\text{either } H\left(g_{t_n} u_{\mathbf{x}} \iota_S \begin{pmatrix} p \\ q \end{pmatrix}\right) \geq \varepsilon \quad \text{or} \quad \frac{d}{dt} \log H\left(g_{t_n} u_{\mathbf{x}} \iota_S \begin{pmatrix} p \\ q \end{pmatrix}\right) \geq 0. \quad (5.5)$$

Proof. If $\mathbf{x} \in \mathbf{BA}_K$, let $\varepsilon = \inf\{\delta_H(g_t \Lambda_{\mathbf{x}}) : t \geq 0\}$; it is positive by Proposition 2.6. Then clearly ε satisfies (5.5) for every sequence t_n . For the converse, since $H\left(g_0 u_{\mathbf{x}} \iota_S \begin{pmatrix} p \\ q \end{pmatrix}\right) \geq 1$ and by (5.4), for every $p, q \in \mathcal{O}$ with $(p, q) \neq (0, 0)$ and for every $t \geq 0$:

$$H\left(g_t u_{\mathbf{x}} \iota_S \begin{pmatrix} p \\ q \end{pmatrix}\right) \geq \min\left\{e^{-dt_1}, \inf_{n \geq 1} \left(\varepsilon e^{-d(t_{n+1} - t_n)}\right)\right\} \geq \min\{e^{-dt_1}, \varepsilon e^{-dc}\} > 0.$$

Hence, $\mathbf{x} \in \mathbf{BA}_K$. □

We end this section with the proof of Theorem 1.3. As usual, we will provide a strategy for Alice, and show that it is indeed a winning strategy for \mathbf{BA}_K .

Proof of Theorem 1.3. We remark that since K_S is a real Banach space, (\mathbf{x}, r) and (\mathcal{L}, ρ) are uniquely defined by $\mathbf{B}(\mathbf{x}, r)$ and $\mathcal{L}^{(\rho)}$ respectively. So to ease the notation, we will identify B_n and A_n with the corresponding sets, and use \mathbf{B}_n and \mathbf{A}_n for this identification in this proof.

Fix $\beta > 0$, and let $\mathbf{B}_n = \mathbf{B}(\mathbf{x}_n, \rho_n)$ be Bob's arbitrary n^{th} move. Without loss of generality, assume that $\rho_1 < 1$ and $\rho_n \rightarrow 0$. Let

$$t_n = -\frac{1}{2} \log(\beta \rho_n), \quad (5.6)$$

and let

$$\varepsilon = 2^{-d} \left(1 + \frac{2}{\beta}\right)^{-d}. \quad (5.7)$$

Then

$$t_{n+1} \leq t_n + \frac{1}{2} \log\left(\frac{1}{\beta}\right) \text{ since } \rho_{n+1} \geq \beta \rho_n. \quad (5.8)$$

If at n^{th} stage of the game, $\delta_H(g_{t_n} \Lambda_{\mathbf{x}}) \geq \varepsilon$ for every $\mathbf{x} \in \mathbf{B}_n$, then Alice can make arbitrary move. Otherwise, let $\mathbf{x} \in \mathbf{B}_n, p_n \in \mathcal{O}, q_n \in \mathcal{O} \setminus \{0\}$ such that:

$$H\left(g_{t_n} u_{\mathbf{x}} \iota_S \begin{pmatrix} p_n \\ q_n \end{pmatrix}\right) < \varepsilon,$$

then by Lemma 5.3 and Remark 5.4, Alice only has to worry about those subspaces in \mathcal{H}_K passing through $\iota_S \begin{pmatrix} p_n \\ q_n \end{pmatrix}$.

For her n^{th} move, Alice will pick the $(\beta \rho_n)$ -neighborhood of all the subspaces in \mathcal{H}_K passing through $\iota_S \begin{pmatrix} p_n \\ q_n \end{pmatrix}$:

$$\mathbf{A}_n = \bigcup_{\#T_{\mathbb{R}} + 2(\#T_{\mathbb{C}}) > \frac{d}{2}} \mathcal{L}\left(\iota_S \begin{pmatrix} p_n \\ q_n \end{pmatrix}, T\right)^{(\beta \rho_n)}. \quad (5.9)$$

Since $\beta\rho_n = e^{-2t_n}$, it follows from (5.3) and Remark 5.4 that, for every $\mathbf{x} \in \mathbf{B}_n \setminus \mathbf{A}_n$ and for every $p, q \in \mathcal{O}$, $(p, q) \neq (0, 0)$:

$$\text{either } H\left(g_{t_n} u_{\mathbf{x}} \iota_S\left(\frac{p}{q}\right)\right) \geq \varepsilon \text{ or } \frac{d}{dt} \log H\left(g_{t_n} u_{\mathbf{x}} \iota_S\left(\frac{p}{q}\right)\right) \geq 0.$$

So $\mathbf{x}_\infty = \bigcap_{n=1}^{\infty} B_n$ satisfies the conditions of Lemma 5.6, and hence, $\mathbf{x}_\infty \in \mathbf{BA}_K$. Thus, by Proposition 4.12, \mathbf{BA}_K is \mathcal{H}_K -absolute winning. \square

APPENDIX A. HATTORI'S APPROACH TO BADLY APPROXIMABLE S -NUMBERS

Hattori [Hat07] proved the following version of Dirichlet's Theorem:

Theorem A.1 ([Hat07, Theorem 1, 2]). *If K be a real quadratic or totally complex quartic number field, then there is a constant $C = C_K > 0$ depending only on K such that for every $\mathbf{x} \in K_S \setminus \iota_S(K)$, there are infinitely many $p \in \mathcal{O}$, $q \in \mathcal{O} \setminus \{0\}$ satisfying:*

$$\left\| \mathbf{x} + \iota_S\left(\frac{p}{q}\right) \right\| \leq \|\iota_S(q)\|^{-2}.$$

Hence, we say that $\mathbf{x} \in K_S \setminus \iota_S(K)$ *badly approximable in Hattori's sense* if there exists $c > 0$ such that for every $p \in \mathcal{O}$, $q \in \mathcal{O} \setminus \{0\}$,

$$\left\| \mathbf{x} + \iota_S\left(\frac{p}{q}\right) \right\| > c \|\iota_S(q)\|^{-2},$$

or equivalently,

$$\inf \left\{ \|\iota_S(q)\|^2 \cdot \left\| \mathbf{x} + \iota_S\left(\frac{p}{q}\right) \right\| : p, q \in \mathcal{O}, q \neq 0 \right\} > 0. \quad (\text{A.1})$$

The set of S -numbers badly approximable in Hattori's sense is denoted by \mathbf{BA}'_K .

Proposition A.2. *Let K be a real quadratic ($d = 2 = \#S$) or totally complex quartic ($d = 4 = 2\#S$) number field; then $\mathbf{BA}_K = \mathbf{BA}'_K$.*

Proof. Recall that $\mathbf{x} \in \mathbf{BA}_K$ if and only if (1.4) holds. Then clearly $\mathbf{BA}_K \subseteq \mathbf{BA}'_K$, since:

$$\begin{aligned} \|\iota_S(q) \cdot \mathbf{x} + \iota_S(p)\| &= \max_{v \in S} |\iota_v(q)x_v + \iota_v(p)| = \max_{v \in S} \left| \iota_v(q) \left(x_v + \iota_v\left(\frac{p}{q}\right) \right) \right| \\ &\leq \max_{v \in S} |\iota_v(q)| \cdot \max_{v \in S} \left| x_v + \iota_v\left(\frac{p}{q}\right) \right| = \|\iota_S(q)\| \cdot \left\| \mathbf{x} + \iota_S\left(\frac{p}{q}\right) \right\|. \end{aligned}$$

For the converse, first by using Lemma 2.5, for every $p \in \mathcal{O}$, $q \in \mathcal{O} \setminus \{0\}$, there exists a unit $\xi \in \mathcal{O}^\times$ such that:

$$|N(q)| \cdot \max_{v \in S} \left| x_v + \iota_v\left(\frac{p}{q}\right) \right|^{d_v} \geq C^{-d} \|\iota_S(\xi q)\|^d \cdot \max_{v \in S} \left| x_v + \iota_v\left(\frac{p}{q}\right) \right|^{d_v} = C^{-d} \left(\|\iota_S(\xi q)\|^2 \cdot \left\| x_v + \iota_v\left(\frac{\xi p}{\xi q}\right) \right\| \right)^{\frac{d}{2}}.$$

In particular, if $\mathbf{x} \in \mathbf{BA}'_K$ then

$$\inf \left\{ |N(q)| \cdot \max_{v \in S} \left| x_v + \iota_v\left(\frac{p}{q}\right) \right|^{d_v} : p, q \in \mathcal{O}, q \neq 0 \right\} > 0. \quad (\text{A.2})$$

Hence, it suffices to show that if $\mathbf{x} \in K_S \setminus \iota_S(K)$ satisfying (A.2), then $\mathbf{x} \in \mathbf{BA}_K$.

Unwrapping Proposition 2.6, we see that $\mathbf{x} \in \mathbf{BA}_K$ if and only if

$$\inf \left\{ \prod_{v \in S} \max \{ |e^{-t} \iota_v(q)|, |e^t(\iota_v(q)x_v + \iota_v(p))| \}^{d_v} : (p, q) \in \mathcal{O}^2 \setminus (0, 0), t \geq 0 \right\} > 0. \quad (\text{A.3})$$

When $q = 0, p \neq 0$,

$$\prod_{v \in S} \max \{ |e^{-t} \iota_v(q)|, |e^t(\iota_v(q)x_v + \iota_v(p))| \}^{d_v} = \prod_{v \in S} |e^t \iota_v(p)|^{d_v} \geq |N(p)| \geq 1.$$

So it suffices to consider the case when $q \neq 0$ and the product in the left hand side of the above formula is < 1 . In that case,

$$\prod_{v \in S} \max \{ |e^{-t} \iota_v(q)|, |e^t(\iota_v(q)x_v + \iota_v(p))| \}^{d_v} = |N(q)| \prod_{v \in S} \max \left\{ e^{-t}, e^t \left| x_v + \iota_v \left(\frac{p}{q} \right) \right| \right\}^{d_v}. \quad (\text{A.4})$$

Since $\#S = 2$ and the d_v 's are both 1's or both 2's,

$$\inf \left\{ \prod_{v \in S} \max \left\{ e^{-t}, e^t \left| x_v + \iota_v \left(\frac{p}{q} \right) \right| \right\}^{d_v} : t \geq 0 \right\} = \max_{v \in S} \left| x_v + \iota_v \left(\frac{p}{q} \right) \right|^{d_v}. \quad (\text{A.5})$$

Combining (A.5) and (A.4), we have the equivalence of (A.3) and (A.2). \square

Remark A.3. Combining the proof of Proposition A.2 with Theorem 2.3 gives us another proof of Theorem A.1 of Hattori.

APPENDIX B. PROOFS OF PROPERTIES OF \mathcal{H} -ABSOLUTE WINNING SETS

Since our definitions are slightly different from those found in [Sch66, BFK⁺12], we provide the proofs of basic properties of \mathcal{H} -absolute winning sets in this appendix for completeness.

B.1. Proof of Lemma 4.7. Let $0 < \beta' \leq \frac{\beta}{2 + \beta}$, and let σ_A be an (\mathcal{H}, β'') -absolute positional winning strategy for Alice for some $0 < \beta'' \leq \beta'$. Since Y is β -diffuse, Lemma 3.4 will guarantee that if Alice is using the σ_A strategy in an (\mathcal{H}, β') -absolute game on Y , Bob will always have eligible moves, and the game will last infinitely. And hence, it is a winning strategy for Alice in the (\mathcal{H}, β') -absolute game. \square

B.2. Proof of Proposition 4.8. Let $0 < \alpha \leq \frac{\beta}{2 + \beta}$ and $0 < \beta' < 1$, and denote $\gamma = \alpha\beta' < \frac{\beta}{2 + \beta}$. By Lemma 4.7, W is (\mathcal{H}, γ) -absolute winning. Let $\sigma_A : Y \times \mathbb{R}_{>0} \rightarrow \mathcal{H} \times \mathbb{R}_{>0}$ be a positional (\mathcal{H}, γ) -absolute winning strategy for Alice. With Remark 4.6 and Lemma 3.4, for any B_n with sufficiently small radius, if we denote $\sigma_A(B_n) = (\mathcal{L}_n, \rho_n)$, Alice can pick a ball $A_n \preceq B_n$ such that:

$$r(A_n) = \alpha r(B_n) \quad \text{and} \quad \text{dist}(c(A_n), \mathcal{L}_n) > 2\alpha r(B_n).$$

Then for any $B_{n+1} \preceq A_n$ with

$$r(B_{n+1}) = \beta' r(A_n) = \beta' \alpha r(B_n) = \gamma r(B_n),$$

B_{n+1} satisfies (H4):

$$\text{dist}(c(B_{n+1}), \mathcal{L}_n) \geq \text{dist}(c(A_n), \mathcal{L}_n) - \alpha(1 - \beta')r(B_n) > 2\gamma r(B_n) \geq r(B_{n+1}) + \rho_n.$$

Therefore, the infinite sequence $B_1, \sigma_A(B_1), B_2, \sigma_A(B_2), \dots$ satisfies (H1)–(H5) for the (\mathcal{H}, γ) -absolute game. That implies:

$$W \cap Y \cap \bigcap_{n=1}^{\infty} \mathbf{B}(B_n) \neq \emptyset.$$

Thus, $W \cap Y$ is $\frac{\beta}{2+\beta}$ -winning. \square

B.3. Proof of Proposition 4.9. We follow the proof of the countable intersection property of Schmidt games [Sch66, Theorem 2]. By part (i) of Lemma 4.7, it suffices to assume that W_1, W_2, \dots be a countable collection of sets that are (\mathcal{H}, γ) -absolute winning on Y with $\gamma = \frac{\beta}{2+\beta}$. For $i = 1, 2, \dots$, let $\sigma_i : Y \times \mathbb{R}_{>0} \rightarrow \mathcal{H} \times \mathbb{R}_{>0}$ be a positional $(\mathcal{H}, \gamma^{2^i})$ -absolute winning strategy for Alice with the target set W_i .

Let $B_1 = (x, \rho) \in Y \times \mathbb{R}_{>0}$ be Bob's arbitrary first move. By Remark 4.6, and Lemma 3.4, Bob always has legitimate moves regardless of Alice's choices following the rules of the (\mathcal{H}, γ) -absolute game.

We define Alice's new strategy σ to be:

$$\sigma(B_1, \dots, B_{2^{i-1}+(n-1)2^i}) = \sigma_i(B_{2^{i-1}+(n-1)2^i}) \quad \text{for } n = 1, 2, 3, \dots$$

It is easy to check that for $i = 1, 2, \dots$, the sequence

$$\begin{aligned} B'_1 &= B_{2^{i-1}}, \\ A'_1 &= \sigma(B_1, B_2, \dots, B_{2^{i-1}}) = \sigma_i(B_{2^{i-1}}) \\ B'_2 &= B_{2^{i-1}+2^i} \\ A'_2 &= \sigma(B_1, B_2, \dots, B_{2^{i-1}+2^i}) = \sigma_i(B_{2^{i-1}+2^i}) \\ &\dots \\ B'_n &= B_{2^{i-1}+(n-1)2^i} \\ A'_n &= \sigma(B_1, B_2, \dots, B_{2^{i-1}+(n-1)2^i}) = \sigma_i(B_{2^{i-1}+(n-1)2^i}) \\ &\dots \end{aligned}$$

satisfies (H1)-(H5) for $(\mathcal{H}, \gamma^{2^i})$ -absolute game, and hence

$$W_i \cap Y \cap \left(\bigcap_{n=1}^{\infty} \mathbf{B}(B_n) \right) = W_i \cap Y \cap \bigcap_{n=1}^{\infty} \mathbf{B}(B_{2^{i-1}+(n-1)2^i}) \neq \emptyset.$$

Since $r(B_n) \rightarrow 0$, the unique intersection point of $\mathbf{B}(B_n)$'s must belongs to all W_i 's. Thus,

$$\left(\bigcap_{i=1}^{\infty} W_i \right) \cap Y \cap \left(\bigcap_{i=1}^{\infty} \mathbf{B}(B_i) \right) \neq \emptyset,$$

and $\bigcap_{i=1}^{\infty} W_i$ is \mathcal{H} -absolute winning. Note that the strategy σ is not a positional winning strategy. \square

B.4. Proof of Proposition 4.10. Let $Z = \{z_1, z_2, \dots\}$, and denote $W_n = W \setminus \{z_1, \dots, z_n\}$. Since $Z \subseteq W \cap \bigcup_{\mathcal{L} \in \mathcal{H}} (\mathcal{L} \cap Y)$, for every $i = 1, 2, \dots$ and for every $\rho_i > 0$, there exists $\mathcal{L}_i \in \mathcal{H}$ such that $z_i \in \mathcal{L}_i^{(\rho_i)}$. If we let $A_i = (\mathcal{L}_i, \rho_i)$ for $1 \leq i \leq n$, and for $i > n$, A_i follow a positional winning strategy for Alice in the \mathcal{H} -absolute game on Y with the target set W , then

$$W_n \cap Y \cap \bigcap_{n=1}^{\infty} \mathbf{B}(B_n) \neq \emptyset.$$

Note that by diffuseness, Bob always has eligible moves. So, W_n is \mathcal{H} -absolute winning on Y for all $n = 1, 2, \dots$. Therefore, by Proposition 4.9,

$$W \setminus Z = \bigcap_{n=1}^{\infty} W_n$$

is \mathcal{H} -absolute winning on Y . \square

B.5. Proof of Proposition 4.12. (i) \Rightarrow (ii): Since $\mathcal{H} \subseteq \mathcal{H}^{*N}$, every choice in (\mathcal{H}, β) -absolute game is available in the $(\mathcal{H}^{*N}, \beta)$ -absolute game. So Alice can apply her (\mathcal{H}, β) -absolute winning strategy in the $(\mathcal{H}^{*N}, \beta)$ -game.

For the converse, assume that Y is (\mathcal{H}, β) -diffuse and consider the (\mathcal{H}, γ) -absolute game, where $\gamma = \frac{\beta}{2 + \beta}$. Let $\sigma : Y \times \mathbb{R}_{>0} \rightarrow \mathcal{H}^{*N} \times \mathbb{R}_{>0}$ be a positional winning strategy for Alice in the $(\mathcal{H}^{*N}, \gamma^N)$ -absolute game on Y . For $k = 0, 1, \dots$, consider Bob's $(kN + 1)^{\text{th}}$ move B_{kN+1} and let

$$\sigma(B_{kN+1}) = (\mathcal{L}_{k,1} \cup \dots \cup \mathcal{L}_{k,N}, \rho_k).$$

Then for $1 \leq i \leq N$, at $(kN + i)^{\text{th}}$ move, Alice chooses

$$A_{kN+i} = (\mathcal{L}_{k,i}, \rho_k).$$

It is easy to check that Alice's choices satisfies (H2), and by Lemma 3.4, Bob can always make a move. To see that this is a winning strategy for (\mathcal{H}, γ) -game, we view the following sequence:

$$B'_1 = B_1, A'_1 = \sigma(B_1), B'_2 = B_{N+1}, A'_2 = \sigma(B_{N+1}), B'_3 = B_{2N+1}, A'_3 = \sigma(B_{2N+1}), \dots$$

as a play in the $(\mathcal{H}^{*N}, \gamma^N)$ -absolute game on Y . It can be verified that this sequence satisfies (H1)–(H5), and since σ is a $(\mathcal{H}^{*N}, \gamma^N)$ -absolute winning strategy,

$$W \cap Y \cap \bigcap_{n=1}^{\infty} \mathbf{B}(B_n) = W \cap Y \cap \bigcap_{n=1}^{\infty} \mathbf{B}(B'_n) \neq \emptyset.$$

Thus, W is \mathcal{H} -absolute winning on Y . \square

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